

Rational Solutions of the $A_4^{(1)}$ Painlevé Equation

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Abstract: We completely classify all of rational solutions of the $A_4^{(1)}$ Painlevé equation, which is a generalization of the fourth Painlevé equation. The rational solutions are classified to three by the Bäcklund transformation group.

Key words: the $A_4^{(1)}$ Painlevé equation; the affine Weyl group; the Bäcklund transformations; rational solutions.

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Introduction

In this paper, we obtain a necessary and sufficient condition for the $A_4^{(1)}$ Painlevé equation to have a rational solution. The $A_4^{(1)}$ Painlevé equation is a generalization of the fourth Painlevé equation. For the classification, we only use the residue calculus. In order to get a necessary condition, we firstly use the residue calculus of a rational solution. By the Bäcklund transformation, we secondly transform the parameters of the $A_4^{(1)}$ Painlevé equation into the fundamental domain. In order to obtain a sufficient condition, we lastly use the residue calculus of the principal part of the Hamiltonian, which is introduced in Section 3.

Paul Painlevé and his pupil [16, 2] classified all differential equations of the form $y'' = F(t, y, y')$ on the complex domain D where F is rational in y, y' , locally analytic in $t \in D$ and for each solution, all the singularities which are dependent on the initial conditions are poles. They found fifty equations of this type, forty four of which can be solved or can be integrated in terms of solutions of ordinary linear differential equations, or elliptic functions. The remaining six equations are called the Painlevé equations and are given by

$$\begin{aligned} P_1 \quad y'' &= 6y^2 + t, \\ P_2 \quad y'' &= 2y^3 + 3ty + \alpha, \\ P_3 \quad y'' &= \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \\ P_4 \quad y'' &= \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \end{aligned}$$

$$\begin{aligned}
P_5 \quad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{1}{t} y' + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \\
P_6 \quad y'' &= \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right),
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters.

Rational solutions of P_J ($J = 2, 3, 4, 5, 6$) were classified by Yablonski and Vorobev [20, 19], Gromak [5, 4], Murata [9], Kitaev, Law and McLeod [6] and Mazzocco [8]. Especially, Murata [9] classified all of rational solutions of the fourth Painlevé equations by using the Bäcklund transformations, which transform a solution into another solution of the same equation with different parameters.

P_J ($J = 2, 3, 4, 5, 6$) have the Bäcklund transformation group. It is shown by Okamoto [12] [13] [14] [15] that the Bäcklund transformation groups are isomorphic to the extended affine Weyl groups. For P_2, P_3, P_4, P_5, P_6 , the Bäcklund transformation groups correspond to $A_1^{(1)}, A_1^{(1)} \oplus A_1^{(1)}, A_2^{(1)}, A_3^{(3)}, D_4^{(1)}$, respectively.

Nowadays, the Painlevé equations are extended in many different ways. Garnier [3] studied isomonodromic deformations of the second order linear equations with many regular singularities. Noumi and Yamada [10] discovered the equations of type $A_l^{(1)}$, whose Bäcklund transformation groups are isomorphic to $\tilde{W}(A_l^{(1)})$. These equations are called the $A_l^{(1)}$ Painlevé equations. The $A_2^{(1)}$ and $A_3^{(1)}$ Painlevé equations correspond to the fourth and fifth Painlevé equations, respectively.

The $A_4^{(1)}$ Painlevé equation is defined by

$$A_4(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) : \begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4 \\ f_0 + f_1 + f_2 + f_3 + f_4 = t, \end{cases}$$

where $'$ is the differentiation with respect to t . For the $A_4^{(1)}$ Painlevé equation, we consider the suffix of f_i and α_i as elements of $\mathbb{Z}/5\mathbb{Z}$. From the $A_4^{(1)}$ Painlevé equation, we have $\sum_{i=0}^4 \alpha_i = 1$. The $A_4^{(1)}$ Painlevé equation is an essentially nonlinear equation with the fourth order. By setting $f_3 \equiv f_4 \equiv 0$, we get the $A_2^{(1)}$ Painlevé equation which is defined

by

$$A_2(\alpha_0, \alpha_1, \alpha_2) : \begin{cases} f'_0 = f_0(f_1 - f_2) + \alpha_0 \\ f'_1 = f_1(f_2 - f_0) + \alpha_1 \\ f'_2 = f_2(f_0 - f_1) + \alpha_2 \\ f_0 + f_1 + f_2 = t, \end{cases}$$

which is equivalent to the forth Painlevé equation. The $A_4^{(1)}$ Painlevé equation is the first equation of the $A_l^{(1)}$ Painlevé equations, which is not the original Painlevé equations. We note that Veselov and Shabat [18], Adler [1] studied the symmetric forms of the Painlevé equations from the viewpoint of soliton.

The Bäcklund transformation group of the $A_4^{(1)}$ Painlevé equation is generated by s_0, s_1, s_2, s_3, s_4 and π :

x	$s_0(x)$	$s_1(x)$	$s_2(x)$	$s_3(x)$	$s_4(x)$	$\pi(x)$
f_0	f_0	$f_0 - \alpha_1/f_1$	f_0	f_0	$f_0 + \alpha_4/f_4$	f_1
f_1	$f_1 + \alpha_0/f_0$	f_1	$f_1 - \alpha_2/f_2$	f_1	f_1	f_2
f_2	f_2	$f_2 + \alpha_1/f_1$	f_2	$f_2 - \alpha_3/f_3$	f_2	f_3
f_3	f_3	f_3	$f_3 + \alpha_2/f_2$	f_3	$f_3 - \alpha_4/f_4$	f_4
f_4	$f_4 - \alpha_0/f_0$	f_4	f_4	$f_4 + \alpha_3/f_3$	f_4	f_0
α_0	$-\alpha_0$	$\alpha_0 + \alpha_1$	α_0	α_0	$\alpha_0 + \alpha_4$	α_1
α_1	$\alpha_1 + \alpha_0$	$-\alpha_1$	$\alpha_1 + \alpha_2$	α_1	α_1	α_2
α_2	α_2	$\alpha_2 + \alpha_1$	$-\alpha_2$	$\alpha_2 + \alpha_3$	α_2	α_3
α_3	α_3	α_3	$\alpha_3 + \alpha_2$	$-\alpha_3$	$\alpha_3 + \alpha_4$	α_4
α_4	$\alpha_4 + \alpha_0$	α_4	α_4	$\alpha_4 + \alpha_3$	$-\alpha_4$	α_0

If $f_i \equiv 0$ for some $i = 0, 1, 2, 3, 4$, we consider s_i as the identical transformation which is given by

$$s_i(f_j) = f_j \text{ and } s_i(\alpha_j) = \alpha_j \text{ } (j = 0, 1, 2, 3, 4).$$

The Bäcklund transformation group $\langle s_0, s_1, s_2, s_3, s_4, \pi \rangle$ is isomorphic to the extended affine Weyl group $\tilde{W}(A_4^{(1)})$.

In this paper, we completely classify rational solutions of the $A_4^{(1)}$ Painlevé equation by using the method of Murata [9]. The result is that rational solutions of the $A_4^{(1)}$ Painlevé equation are decomposed to three classes, each of which is an orbit by the action of $\tilde{W}(A_4^{(1)})$.

This paper is organized as follows. Section 1 consists of two subsections. In Subsection 1.1, we calculate the Laurent series of a rational solution $(f_i)_{0 \leq i \leq 4}$ of $A_4(\alpha_i)_{0 \leq i \leq 4}$ at $t = \infty$. The residues of f_i ($0 \leq i \leq 4$) are expressed by the parameters α_i ($0 \leq i \leq 4$). In Proposition 1.1, 1.2, 1.3, we determine the Laurent series of f_i ($0 \leq i \leq 4$) of $A_4(\alpha_i)_{0 \leq i \leq 4}$

and obtain a sufficient condition for f_i ($0 \leq i \leq 4$) to be uniquely expanded at $t = \infty$. In Subsection 1.2, we get the Laurent series of a rational solution $(f_i)_{0 \leq i \leq 4}$ of $A_4(\alpha_i)_{0 \leq i \leq 4}$ at $t = c \in \mathbb{C}$ following Tahara [17].

In Section 2, we firstly introduce shift operators, following Noumi and Yamada [11]. Secondly, from the residue theorem, we get a necessary condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution and prove that if $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution, the parameters α_i ($0 \leq i \leq 4$) are rational numbers. In addition, we transform the parameters into the set C which is defined by

$$C := \{(\alpha_i)_{0 \leq i \leq 4} \in \mathbb{R}^5 \mid 0 \leq \alpha_i \leq 1 \ (0 \leq i \leq 4)\}.$$

In Section 3, we firstly introduce the Hamiltonian H of $A_4(\alpha_i)_{0 \leq i \leq 4}$ and its principal part \hat{H} following Noumi and Yamada [11]. Secondly, we calculate the residues of \hat{H} at $t = \infty, c$ and prove Lemma 3.3, which is devoted to the residue calculus of \hat{H} . We use Lemma 3.3 in order to obtain a sufficient condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution. Thirdly, with the residue calculus of \hat{H} , we prove Theorem 0.1 which gives us a necessary and sufficient condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution.

The main result of this paper was announced in [7].

Theorem 0.1. *The $A_4^{(1)}$ Painlevé equation has a rational solution if and only if the parameters α_j ($0 \leq j \leq 4$) satisfy one of the following three conditions. The solution is unique, if it exists.*

- (1) $\alpha_0, \alpha_1, \dots, \alpha_4 \in \mathbb{Z}$.
- (2) For some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \pm \frac{1}{3}(1, 1, 1, 0, 0) & \text{mod } \mathbb{Z} \\ \pm \frac{1}{3}(1, -1, -1, 1, 0) & \text{mod } \mathbb{Z}. \end{cases}$$

- (3) For some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \frac{j}{5}(1, 1, 1, 1, 1) & \text{mod } \mathbb{Z} \\ \frac{j}{5}(1, 2, 1, 3, 3) & \text{mod } \mathbb{Z}, \end{cases}$$

with some $j = 1, 2, 3, 4$.

- (4) Furthermore, by a suitable Bäcklund transformation, the rational solution in the class (1), (2), (3) above is respectively transformed into the following.

(i)

$$(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0, 0),$$

(ii)

$$(f_0, f_1, f_2, f_3, f_4) = \left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0\right) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right),$$

(iii)

$$(f_0, f_1, f_2, f_3, f_4) = \left(\frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}\right) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

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1 The Expansions of Rational Solutions

This section consists of two subsections. In Subsection 1.1, we suppose that $(f_j)_{0 \leq j \leq 4}$ is a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$. We calculate the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty, c \in \mathbb{C}$. The residues of f_j ($0 \leq j \leq 4$) at $t = \infty$ are expressed by the parameters α_j ($0 \leq j \leq 4$) and the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ are uniquely expanded under the conditions in Proposition 1.3.

In Subsection 1.2, following Tahara [17], we compute the residues of f_j ($0 \leq j \leq 4$) at $t = c \in \mathbb{C}$, which are integers.

1.1 the Laurent Series at $t = \infty$

In this subsection, we prove Proposition 1.1, 1.2 and 1.3. In Proposition 1.1, we determine the order of a pole of f_i ($0 \leq i \leq 4$) at $t = \infty$. In Proposition 1.2, we get the residues of $(f_i)_{0 \leq i \leq 4}$ at $t = \infty$. In Proposition 1.3, we obtain a sufficient condition for the Laurent series of f_i ($0 \leq i \leq 4$) at $t = \infty$ to be uniquely expanded.

Proposition 1.1. *Suppose that $(f_0, f_1, f_2, f_3, f_4)$ is a rational solution of $A_4(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and some of f_0, f_1, f_2, f_3, f_4 have a pole at $t = \infty$. Then $(f_0, f_1, f_2, f_3, f_4)$ satisfies one of the following conditions:*

- (1) *for some $i = 0, 1, 2, 3, 4$, f_i has a pole at $t = \infty$ with the first order;*
- (2) *for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$ with the first order;*
- (3) *for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$ with the first order;*
- (4) *all of f_0, f_1, f_2, f_3, f_4 have a pole at $t = \infty$ with the first order.*

We denote the case (1) by Type A (1), the case (2) by Type A (2), the case (3) by Type B and the case (4) by Type C, respectively.

Proof. We set

$$\begin{cases} f_0 = \sum_{k=-\infty}^{n_0} a_k t^k, & f_1 = \sum_{k=-\infty}^{n_1} b_k t^k, & f_2 = \sum_{k=-\infty}^{n_2} c_k t^k, \\ f_3 = \sum_{k=-\infty}^{n_3} d_k t^k, & f_4 = \sum_{k=-\infty}^{n_4} e_k t^k, \end{cases} \quad (1.1)$$

where n_0, n_1, n_2, n_3, n_4 are integers.

Since $\sum_{k=0}^4 f_k = t$, the following five cases occur.

- I one rational function of $(f_k)_{0 \leq k \leq 4}$ has a pole at $t = \infty$,
- II two rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$,
- III three rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$,
- IV four rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$,
- V all the rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$.

Case I: one of rational function $(f_k)_{0 \leq k \leq 4}$ has a pole at $t = \infty$. By π , we assume that f_0 has a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$n_0 = 1, n_j \leq 0 \quad (1 \leq j \leq 4).$$

Therefore, we get Type A (1).

Case II: two rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$. Since the suffix of f_i and α_i are considered as elements of $\mathbb{Z}/5\mathbb{Z}$, the following two cases occur.

- (1) for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1} have a pole at $t = \infty$,
- (2) for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+2} have a pole at $t = \infty$.

Case II (1): f_i, f_{i+1} have a pole at $t = \infty$. By π , we assume that f_0, f_1 have a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$n_0 = n_1 \geq 1, n_j \leq 0 \quad (j = 2, 3, 4).$$

By comparing the highest terms in

$$f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0,$$

we obtain

$$n_0 - 1 = 2n_0.$$

Therefore, we have $n_0 = -1$, which contradiction.

Case II (2): f_i, f_{i+2} have a pole at $t = \infty$. By π , we assume that f_0, f_2 have a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$n_0 = n_2 \geq 1, \quad n_j \leq 0 \quad (j = 1, 3, 4).$$

By comparing the highest terms in

$$f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0,$$

we obtain

$$n_0 - 1 = 2n_0.$$

Therefore, we have $n_0 = -1$, which is contradiction.

Case III: three rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$. Since the suffix of f_i and α_i are considered as elements of $\mathbb{Z}/5\mathbb{Z}$, the following two cases occur.

- (1) for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$.
- (2) for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$.

Case III (1): f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$. By π , we assume that f_0, f_1, f_2 have a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, the following four cases occur.

- (i) $n_0 = n_1 > n_2 \geq 1$
- (ii) $n_1 = n_2 > n_0 \geq 1$
- (iii) $n_2 = n_0 > n_1 \geq 1$
- (iv) $n_0 = n_1 = n_2 \geq 1$.

Case III (1) (i): $n_0 = n_1 > n_2 \geq 1$. By comparing the highest terms in

$$f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0,$$

we have

$$n_0 - 1 = 2n_0.$$

Therefore, we have $n_0 = -1$, which is contradiction.

Case III (1) (ii) and (iii): $n_1 = n_2 > n_0 \geq 1$ or $n_2 = n_0 > n_1 \geq 1$. We can show contradiction in the same way.

Case III (1) (iv): $n_0 = n_1 = n_2 \geq 1$. By comparing the highest terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2, \end{cases}$$

we have

$$\begin{cases} b_{n_1} - c_{n_2} = 0 \\ c_{n_2} - a_{n_0} = 0 \\ a_{n_0} - b_{n_1} = 0. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$n_0 = n_1 = n_2 = 1, \quad a_1 = b_1 = c_1 = \frac{1}{3}.$$

Therefore, we get Type B.

Case III (2): f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$. By π , we assume that f_0, f_1, f_3 have a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, the following four cases occur.

$$\begin{array}{ll} \text{(i)} & n_0 = n_1 > n_3, \\ \text{(ii)} & n_1 = n_3 > n_0, \\ \text{(iii)} & n_3 = n_1 > n_0, \\ \text{(iv)} & n_0 = n_1 = n_3. \end{array}$$

If the cases III (1) (i), (ii) and (iii) occur, we can show contradiction in the same way as the case II.

Case III (iv): $n_0 = n_1 = n_3$. We suppose that $n_0 = n_1 = n_3 \geq 2$. By comparing the highest terms in

$$f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0,$$

we get

$$b_{n_1} + d_{n_3} = 0.$$

Since $\sum_{k=0}^4 f_k = t$, it follows that $a_{n_0} = 0$, which is contradiction. Therefore, we obtain

$$n_0 = n_1 = n_3 = 1$$

and get Type A (2).

Case IV: four rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$. By π , we

assume that f_0, f_1, f_2, f_3 have a pole at $t = \infty$. Then the following eleven cases occur.

- | | |
|--|---|
| (i) $n_0 = n_1 > \left\{ \begin{smallmatrix} n_2 \\ n_3 \end{smallmatrix} \right\} \geq 1$ | (ii) $n_0 = n_2 > \left\{ \begin{smallmatrix} n_1 \\ n_3 \end{smallmatrix} \right\} \geq 1$ |
| (iii) $n_0 = n_3 > \left\{ \begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix} \right\} \geq 1$ | (iv) $n_1 = n_2 > \left\{ \begin{smallmatrix} n_0 \\ n_3 \end{smallmatrix} \right\} \geq 1$ |
| (v) $n_1 = n_3 > \left\{ \begin{smallmatrix} n_0 \\ n_2 \end{smallmatrix} \right\} \geq 1$ | (vi) $n_2 = n_3 > \left\{ \begin{smallmatrix} n_0 \\ n_1 \end{smallmatrix} \right\} \geq 1$ |
| (vii) $n_0 = n_1 = n_2 > n_3 \geq 1$ | (viii) $n_0 = n_1 = n_3 > n_2 \geq 1$ |
| (ix) $n_1 = n_2 = n_3 > n_0 \geq 1$ | (x) $n_2 = n_3 = n_0 > n_1 \geq 1$ |
| (xi) $n_0 = n_1 = n_2 = n_3 \geq 1$. | |

If the cases IV (i), (ii), \dots , (vi) occur, we can show contradiction in the same way as the case II.

Case IV (vii) or (ix): $n_0 = n_1 = n_2 > n_3 \geq 1$ or $n_1 = n_2 = n_3 > n_0 \geq 1$. We deal with the case IV (vii). The case IV (ix) can be proved in the same way. By comparing the highest terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \end{cases}$$

we get

$$\begin{cases} b_{n_1} - c_{n_2} = 0 \\ c_{n_2} - a_{n_0} = 0. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_{n_0} = b_{n_1} = c_{n_2} = 0,$$

which is contradiction.

Case IV (viii) or (x): $n_0 = n_1 = n_3 > n_2 \geq 1$ or $n_2 = n_3 = n_0 > n_1 \geq 1$. We deal with the case IV (viii). The case IV (x) can be proved in the same way. By comparing the highest terms in

$$f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2,$$

we have

$$d_{n_3} + a_{n_0} - b_{n_1} = 0.$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$b_{n_1} = 0,$$

which is contradiction.

Case IV (xi): $n_0 = n_1 = n_2 = n_3 \geq 1$. By comparing the highest terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3, \end{cases}$$

we obtain

$$b_{n_1} - c_{n_2} + d_{n_3} = 0 \tag{1.2}$$

$$c_{n_2} - d_{n_3} - a_{n_3} = 0 \tag{1.3}$$

$$d_{n_3} + a_{n_0} - b_{n_1} = 0 \tag{1.4}$$

$$-a_{n_0} + b_{n_1} - c_{n_2} = 0. \tag{1.5}$$

We assume that $n_0 = n_1 = n_2 = n_3 \geq 2$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_{n_0} = -2c_{n_2}, b_{n_1} = c_{n_2}, d_{n_3} = 3c_{n_2}.$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$c_{n_2} = 0,$$

which is contradiction.

We assume that $n_0 = n_1 = n_2 = n_3 = 1$. The equation (1.2) implies that

$$a_1 + 2c_1 = 1,$$

because $\sum_{k=0}^4 f_k = t$. The equations (1.3) and (1.4) imply that

$$d_1 = 3c_1 - 1, \quad b_1 = c_1.$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$1 = a_1 + b_1 + c_1 + d_1 = 3c_1.$$

Therefore we obtain

$$c_1 = \frac{1}{3}, \quad d_1 = 0,$$

which is contradiction.

Case VI: all the rational functions of $(f_k)_{0 \leq k \leq 4}$ have a pole at $t = \infty$. Since $\sum_{k=0}^4 f_k = t$, the following twelve cases occur.

$$\begin{aligned}
\text{(i)} \quad n_0 = n_1 &> \begin{Bmatrix} n_2 \\ n_3 \\ n_4 \end{Bmatrix} \geq 1, & \text{(ii)} \quad n_0 = n_2 &> \begin{Bmatrix} n_1 \\ n_3 \\ n_4 \end{Bmatrix} \geq 1, \\
\text{(iii)} \quad n_0 = n_3 &> \begin{Bmatrix} n_1 \\ n_2 \\ n_4 \end{Bmatrix} \geq 1, & \text{(iv)} \quad n_0 = n_4 &> \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \geq 1, \\
\text{(v)} \quad n_0 = n_1 = n_2 &> \begin{Bmatrix} n_3 \\ n_4 \end{Bmatrix} \geq 1, & \text{(vi)} \quad n_0 = n_1 = n_3 &> \begin{Bmatrix} n_2 \\ n_4 \end{Bmatrix} \geq 1, \\
\text{(vii)} \quad n_0 = n_1 = n_4 &> \begin{Bmatrix} n_2 \\ n_3 \end{Bmatrix} \geq 1, & \text{(viii)} \quad n_0 = n_2 = n_3 &> \begin{Bmatrix} n_1 \\ n_4 \end{Bmatrix} \geq 1, \\
\text{(ix)} \quad n_0 = n_2 = n_4 &> \begin{Bmatrix} n_1 \\ n_3 \end{Bmatrix} \geq 1, & \text{(x)} \quad n_0 = n_3 = n_4 &> \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} \geq 1, \\
\text{(xi)} \quad n_0 = n_1 = n_2 = n_3 &> n_4 \geq 1, & \text{(xii)} \quad n_0 = n_1 = n_2 = n_3 = n_4 &\geq 1.
\end{aligned}$$

If the cases VI (i), ..., (iv) occur, we can prove contradiction in the same way as the case II. If the cases VI (v), ..., (x) occur, we can prove contradiction in the same way as the case III.

Case VI (xi): $n_0 = n_1 = n_2 = n_3 > n_4 \geq 1$. By comparing the highest terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we have

$$b_{n_1} - c_{n_2} + d_{n_3} = 0 \quad (1.6)$$

$$c_{n_2} - d_{n_3} - a_{n_0} = 0 \quad (1.7)$$

$$d_{n_3} + a_{n_0} - b_{n_1} = 0 \quad (1.8)$$

$$-a_{n_0} + b_{n_1} - c_{n_2} = 0 \quad (1.9)$$

$$a_{n_0} - b_{n_1} + c_{n_2} - d_{n_3} = 0. \quad (1.10)$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_{n_0} + b_{n_1} + c_{n_2} + d_{n_3} = 0. \quad (1.11)$$

The equations (1.6) and (1.11) imply that

$$a_{n_0} = -2c_{n_2}.$$

The equations (1.7) and (1.8) imply that

$$d_{n_3} = 3c_{n_2}, \quad b_{n_1} = c_{n_2}.$$

The equation (1.11) implies that $c_{n_2} = 0$, which is contradiction.

Case VI (xii): $n_0 = n_1 = n_2 = n_3 = n_4 \geq 1$. By comparing the highest terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we obtain

$$\begin{cases} b_{n_1} - c_{n_2} + d_{n_3} - e_{n_4} = 0 \\ c_{n_2} - d_{n_3} + e_{n_4} - a_{n_0} = 0 \\ d_{n_3} - e_{n_4} + a_{n_0} - b_{n_1} = 0 \\ e_{n_4} - a_{n_0} + b_{n_1} - c_{n_2} = 0 \\ a_{n_0} - b_{n_1} + c_{n_2} - d_{n_3} = 0. \end{cases}$$

Since the rank of

$$\begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

is four, it follows that

$$(a_{n_0}, b_{n_1}, c_{n_2}, d_{n_3}, e_{n_4}) = \alpha (1, 1, 1, 1, 1),$$

for some $\alpha \in \mathbb{C}^*$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$n_0 = n_1 = n_2 = n_3 = n_4 = 1, \quad a_1 = b_1 = c_1 = d_1 = e_1 = \frac{1}{5}.$$

Therefore, we get Type C. □

In the following proposition, we obtain the residues of f_i ($0 \leq i \leq 4$) at $t = \infty$ for $A(\alpha_i)_{0 \leq i \leq 4}$. The residues of f_i ($0 \leq i \leq 4$) at $t = \infty$ are expressed by the parameters α_j ($0 \leq j \leq 4$).

Proposition 1.2. *Suppose that $(f_j)_{0 \leq j \leq 4}$ is a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$.*

(1) *If f_i has a pole at $t = \infty$ for some $i = 0, 1, 2, 3, 4$,*

$$\begin{cases} f_i = t + (-\alpha_{i+1} + \alpha_{i+2} - \alpha_{i+3} + \alpha_{i+4})t^{-1} + \dots \\ f_{i+1} = \alpha_{i+1}t^{-1} + \dots \\ f_{i+2} = -\alpha_{i+2}t^{-1} + \dots \\ f_{i+3} = \alpha_{i+3}t^{-1} + \dots \\ f_{i+4} = -\alpha_{i+4}t^{-1} + \dots \end{cases}$$

(2) *If f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$ for some $i = 0, 1, 2, 3, 4$,*

$$\begin{cases} f_i = t + (1 - \alpha_i)t^{-1} + \dots \\ f_{i+1} = t + (1 - \alpha_{i+1} - 2\alpha_{i+2} + 2\alpha_{i+4})t^{-1} + \dots \\ f_{i+2} = \alpha_{i+2}t^{-1} + \dots \\ f_{i+3} = -t + (-1 - \alpha_{i+3} - 2\alpha_{i+4})t^{-1} + \dots \\ f_{i+4} = -\alpha_{i+4}t^{-1} + \dots \end{cases}$$

(3) *If f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$ for some $i = 0, 1, 2, 3, 4$,*

$$\begin{cases} f_i = \frac{1}{3}t + (\alpha_{i+1} - \alpha_{i+2} - 3\alpha_{i+3} - \alpha_{i+4})t^{-1} + \dots \\ f_{i+1} = \frac{1}{3}t + (\alpha_{i+2} - \alpha_i - \alpha_{i+3} + \alpha_{i+4})t^{-1} + \dots \\ f_{i+2} = \frac{1}{3}t + (\alpha_i - \alpha_{i+1} + \alpha_{i+3} + 3\alpha_{i+4})t^{-1} + \dots \\ f_{i+3} = 3\alpha_{i+3}t^{-1} + \dots \\ f_{i+4} = -3\alpha_{i+4}t^{-1} + \dots \end{cases}$$

(4) *If all the rational functions of $(f_0, f_1, f_2, f_3, f_4)$ have a pole at $t = \infty$,*

$$\begin{cases} f_0 = \frac{1}{5}t + (3\alpha_1 + \alpha_2 - \alpha_3 - 3\alpha_4)t^{-1} + \dots \\ f_1 = \frac{1}{5}t + (3\alpha_2 + \alpha_3 - \alpha_4 - 3\alpha_0)t^{-1} + \dots \\ f_2 = \frac{1}{5}t + (3\alpha_3 + \alpha_4 - \alpha_0 - 3\alpha_1)t^{-1} + \dots \\ f_3 = \frac{1}{5}t + (3\alpha_4 + \alpha_0 - \alpha_1 - 3\alpha_2)t^{-1} + \dots \\ f_4 = \frac{1}{5}t + (3\alpha_0 + \alpha_1 - \alpha_2 - 3\alpha_3)t^{-1} + \dots \end{cases}$$

Proof. Type A (1): for some $i = 0, 1, 2, 3, 4$, f_i has a pole at $t = \infty$. By π , we assume that f_0 has a pole at $t = \infty$. Then it follows from Proposition 1.1 that

$$\begin{cases} f_0 = \sum_{k=-\infty}^1 a_k t^k, & f_1 = \sum_{k=-\infty}^{n_1} b_k t^k, & f_2 = \sum_{k=-\infty}^{n_2} c_k t^k, \\ f_3 = \sum_{k=-\infty}^{n_3} d_k t^k, & f_4 = \sum_{k=-\infty}^{n_4} e_k t^k, \end{cases}$$

where $n_1, n_2, n_3, n_4 \leq 0$. Since $\sum_{k=0}^4 f_k = t$, it follows that $a_1 = 1$. By comparing the coefficients of the term t^{n_1+1} in

$$f_1' = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1,$$

we get

$$n_1 = -1, \quad b_{-1} = \alpha_1, \quad \text{or } f_1 \equiv 0.$$

In the same way, we obtain

$$\begin{aligned} n_2 &= -1, \quad c_{-1} = -\alpha_2, \quad \text{or } f_2 \equiv 0, \\ n_3 &= -1, \quad d_{-1} = \alpha_3, \quad \text{or } f_3 \equiv 0, \\ n_4 &= -1, \quad e_{-1} = -\alpha_4, \quad \text{or } f_4 \equiv 0. \end{aligned}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_0 = 0, \quad a_{-1} = -\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4.$$

Type A (2): for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$. By π , we assume that f_0, f_1, f_3 have a pole at $t = \infty$. Then it follows from Proposition 1.1 that

$$\begin{cases} f_0 = \sum_{k=-\infty}^1 a_k t^k, & f_1 = \sum_{k=-\infty}^1 b_k t^k, & f_2 = \sum_{k=-\infty}^{n_2} c_k t^k, \\ f_3 = \sum_{k=-\infty}^1 d_k t^k, & f_4 = \sum_{k=-\infty}^{n_4} e_k t^k, \end{cases} \quad (1.12)$$

where $n_2, n_4 \leq 0$. By comparing the coefficients of the term t^2 in

$$\begin{cases} f_0' = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f_1' = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \end{cases}$$

we have

$$\begin{cases} b_1 + d_1 = 0 \\ a_1 + d_1 = 0. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_1 = b_1 = 1, \quad d_1 = -1.$$

By comparing the coefficients of the term t in

$$\begin{cases} f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we obtain

$$c_0 = e_0 = 0.$$

By comparing the constant terms in

$$\begin{cases} f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we have

$$c_{-1} = \alpha_2, \quad e_{-1} = -\alpha_4.$$

By comparing the coefficients of the term t in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \end{cases}$$

we obtain

$$\begin{cases} b_0 + d_0 = 0 \\ a_0 + d_0 = 0. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_0 = b_0 = d_0 = 0.$$

By comparing the constant terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \end{cases}$$

we have

$$\begin{cases} a_{-1} = -2\alpha_2 + 2\alpha_4 + \alpha_0 - 1 \\ d_{-1} = -\alpha_0 + \alpha_1 + 3\alpha_2 - 3\alpha_4. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$b_{-1} = -\alpha_1 - 2\alpha_2 + 2\alpha_4 + 1.$$

Type B: for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$. By π , we assume that f_0, f_1, f_2 have a pole at $t = \infty$. Then it follows from Proposition 1.1 and its proof that

$$\begin{cases} f_0 = \frac{1}{3}t + \sum_{k=-\infty}^0 a_k t^k, & f_1 = \frac{1}{3}t + \sum_{k=-\infty}^0 b_k t^k, & f_2 = \frac{1}{3}t + \sum_{k=-\infty}^0 c_k t^k, \\ f_3 = \sum_{k=-\infty}^{n_3} d_k t^k, & f_4 = \sum_{k=-\infty}^{n_4} e_k t^k, \end{cases} \quad (1.13)$$

where $n_3, n_4 \leq 0$. By comparing the coefficients of the term t in

$$f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3,$$

we obtain $d_0 = 0$. By comparing the constant terms in

$$f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3,$$

we have

$$d_{-1} = 3\alpha_3.$$

In the same way, we get

$$e_0 = 0, \quad e_{-1} = -3\alpha_4.$$

By comparing the coefficients of the term t in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \end{cases}$$

we obtain

$$\begin{cases} b_0 - a_0 = 0 \\ c_0 - a_0 = 0. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_0 = b_0 = c_0 = 0.$$

By comparing the constant terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2, \end{cases}$$

we get

$$\begin{cases} b_{-1} - c_{-1} = 1 - 3\alpha_0 - 3\alpha_3 - 3\alpha_4 \\ c_{-1} - a_{-1} = 1 - 3\alpha_1 + 3\alpha_3 + 3\alpha_4 \\ a_{-1} - b_{-1} = 1 - 3\alpha_2 - 3\alpha_3 - 3\alpha_4. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$\begin{cases} a_{-1} = \alpha_1 - \alpha_2 - 3\alpha_3 - \alpha_4 \\ b_{-1} = -\alpha_0 + \alpha_2 - \alpha_3 + \alpha_4 \\ c_{-1} = \alpha_0 - \alpha_1 + \alpha_3 + 3\alpha_4. \end{cases}$$

Type C: all the rational functions of f_0, f_1, f_2, f_3, f_4 have a pole at $t = \infty$. Then it follows from Proposition 1.1 and its proof that

$$\begin{cases} f_0 = \frac{1}{5}t + \sum_{k=-\infty}^0 a_k t^k, & f_1 = \frac{1}{5}t + \sum_{k=-\infty}^0 b_k t^k, & f_2 = \frac{1}{5}t + \sum_{k=-\infty}^0 c_k t^k, \\ f_3 = \frac{1}{5}t + \sum_{k=-\infty}^0 d_k t^k, & f_4 = \frac{1}{5}t + \sum_{k=-\infty}^0 e_k t^k. \end{cases} \quad (1.14)$$

By comparing the coefficients of the term t in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we obtain

$$\begin{cases} b_0 - c_0 + d_0 - e_0 = 0 \\ c_0 - d_0 + e_0 - a_0 = 0 \\ d_0 - e_0 + a_0 - b_0 = 0 \\ e_0 - a_0 + b_0 - c_0 = 0 \\ a_0 - b_0 + c_0 - d_0 = 0. \end{cases}$$

Since the rank of

$$\begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

is four, it follows that

$$(a_0, b_0, c_0, d_0, e_0) = \beta (1, 1, 1, 1, 1),$$

for some $\beta \in \mathbb{C}$. Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_0 = b_0 = c_0 = d_0 = e_0 = 0.$$

By comparing the constant terms in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we obtain

$$\begin{cases} 1 = b_{-1} - c_{-1} + d_{-1} + e_{-1} + 5\alpha_0 \\ 1 = c_{-1} - d_{-1} + e_{-1} - a_{-1} + 5\alpha_1 \\ 1 = d_{-1} - e_{-1} + a_{-1} - b_{-1} + 5\alpha_2 \\ 1 = e_{-1} - a_{-1} + b_{-1} - c_{-1} + 5\alpha_3 \\ 1 = a_{-1} - b_{-1} + c_{-1} - d_{-1} + 5\alpha_4. \end{cases}$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_{-1} + b_{-1} + c_{-1} + d_{-1} + e_{-1} = 0.$$

Therefore we get

$$\begin{cases} a_{-1} = 3\alpha_1 + \alpha_2 - \alpha_3 - 3\alpha_4, \\ b_{-1} = 3\alpha_2 + \alpha_3 - \alpha_4 - 3\alpha_0, \\ c_{-1} = 3\alpha_3 + \alpha_4 - \alpha_0 - 3\alpha_1, \\ d_{-1} = 3\alpha_4 + \alpha_0 - \alpha_1 - 3\alpha_2, \\ e_{-1} = 3\alpha_0 + \alpha_1 - \alpha_2 - 3\alpha_3. \end{cases}$$

□

In the following proposition, we get a sufficient condition for the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ to be uniquely expanded.

Proposition 1.3. *Suppose that $(f_j)_{0 \leq j \leq 4}$ is a rational solution on Proposition 1.2.*

(1) *If f_i has a pole at $t = \infty$ and $f_{i+1}, f_{i+2}, f_{i+3}, f_{i+4}$ are regular at $t = \infty$ for some $i = 0, 1, 2, 3, 4$, the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ are uniquely expanded.*

(2) *If f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$ and f_{i+2}, f_{i+4} are regular at $t = \infty$ for some $i = 0, 1, 2, 3, 4$, the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ are uniquely expanded.*

(3) *If f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$ and f_{i+3}, f_{i+4} are regular at $t = \infty$ for some $i = 0, 1, 2, 3, 4$, the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ are uniquely expanded.*

(4) *If all the rational functions of $(f_i)_{0 \leq i \leq 4}$ have a pole at $t = \infty$, the Laurent series of f_j ($0 \leq j \leq 4$) at $t = \infty$ are uniquely expanded.*

Epecially, we have the following:

Type A (1): for some $i = 0, 1, 2, 3, 4$, f_i has a pole at $t = \infty$ and $f_{i+1}, f_{i+2}, f_{i+3}, f_{i+4}$ are regular at $t = \infty$. Then,

$$\begin{cases} f_{i+1} \equiv 0 \text{ if } \alpha_{i+1} = 0 \\ f_{i+2} \equiv 0 \text{ if } \alpha_{i+2} = 0 \\ f_{i+3} \equiv 0 \text{ if } \alpha_{i+3} = 0 \\ f_{i+4} \equiv 0 \text{ if } \alpha_{i+4} = 0. \end{cases}$$

Type A (2): for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$ and f_{i+2}, f_{i+4} are regular at $t = \infty$. Then,

$$\begin{cases} f_{i+2} \equiv 0 \text{ if } \alpha_{i+2} = 0 \\ f_{i+4} \equiv 0 \text{ if } \alpha_{i+4} = 0. \end{cases}$$

Type B: for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$ and f_{i+3}, f_{i+4} are regular at $t = \infty$. Then,

$$\begin{cases} f_{i+3} \equiv 0 \text{ if } \alpha_{i+3} = 0 \\ f_{i+4} \equiv 0 \text{ if } \alpha_{i+4} = 0. \end{cases}$$

Proof. If there exists a rational solution of Type A (1), we have

$$\begin{cases} f_0 = t + a_{-1}t^{-1} + \sum_{k=-\infty}^{-2} a_k t^k, & f_1 = b_{-1}t^{-1} + \sum_{k=-\infty}^{-2} b_k t^k, & f_2 = c_{-1}t^{-1} + \sum_{k=-\infty}^{-2} c_k t^k, \\ f_3 = d_{-1}t^{-1} + \sum_{k=-\infty}^{-2} d_k t^k, & f_4 = e_{-1}t^{-1} + \sum_{k=-\infty}^{-2} e_k t^k, \end{cases}$$

where $a_{-1}, b_{-1}, c_{-1}, d_{-1}, e_{-1}$ have been determined in Proposition 1.2. By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we get

$$\begin{cases} b_{k-1} = b_k(k+1) + \sum_{m=k}^0 b_{k-m}(c_m - d_m + e_m - a_m) \\ c_{k-1} = -c_k(k+1) - \sum_{m=k}^0 c_{k-m}(d_m - e_m + a_m - b_m) \\ d_{k-1} = d_{k+1}(k+1) + \sum_{m=k}^0 d_{k-m}(e_m - a_m + b_m - c_m) \\ e_{k-1} = -e_{k+1}(k+1) - \sum_{m=k}^0 e_{k-m}(a_m - e_m + c_m - d_m). \end{cases}$$

In Proposition 1.2, we have had

$$b_{-1} = \alpha_1, c_{-1} = -\alpha_2, d_{-1} = \alpha_3, e_{-1} = -\alpha_4.$$

Therefore we get

$$\begin{cases} f_1 \equiv 0 \text{ if } \alpha_1 = 0 \\ f_2 \equiv 0 \text{ if } \alpha_2 = 0 \\ f_3 \equiv 0 \text{ if } \alpha_3 = 0 \\ f_4 \equiv 0 \text{ if } \alpha_4 = 0. \end{cases}$$

Since $\sum_{j=0}^4 f_j = t$, it follows that

$$a_{k-1} = -b_{k-1} - c_{k-1} - d_{k-1} - e_{k-1}.$$

Therefore, if there is a rational solution of Type A (1), the coefficients a_k, b_k, c_k, d_k, e_k ($k \leq -2$) are determined inductively and it is unique.

If there exists a rational solution of Type A (2), we have

$$\begin{cases} f_0 = t + a_{-1}t^{-1} + \sum_{k=-\infty}^{-2} a_k t^k, & f_1 = t + b_{-1}t^{-1} + \sum_{k=-\infty}^{-2} b_k t^k, & f_2 = c_{-1}t^{-1} + \sum_{k=-\infty}^{-2} c_k t^k, \\ f_3 = -t + d_{-1}t^{-1} + \sum_{k=-\infty}^{-2} d_k t^k, & f_4 = e_{-1}t^{-1} + \sum_{k=-\infty}^{-2} e_k t^k, \end{cases}$$

where $a_{-1}, b_{-1}, c_{-1}, d_{-1}, e_{-1}$ have been determined in Proposition 1.2. By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we get

$$\begin{cases} c_{k-1} = c_{k+1}(k+1) + \sum_{m=k}^0 c_{(k-m)}(d_m - e_m + a_m - b_m) \\ e_{k-1} = -e_{k+1}(k+1) - \sum_{m=k}^0 e_{k-m}(a_m - e_m + c_m - d_m). \end{cases}$$

In Proposition 1.2, we have had

$$c_{-1} = \alpha_2, \quad e_{-1} = -\alpha_4.$$

Therefore the coefficients c_k, e_k ($k \leq -2$) are determined inductively and we get

$$\begin{cases} f_2 \equiv 0 \text{ if } \alpha_2 = 0 \\ f_4 \equiv 0 \text{ if } \alpha_4 = 0 \end{cases}$$

By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3, \end{cases}$$

we get

$$\begin{cases} b_{k-1} + d_{k-1} = c_{k-1} + e_{k-1} - a_{k+1}(k+1) \\ \quad - \sum_{m=k}^0 a_{(k-m)}(b_m - c_m + d_m - e_m) \\ -d_{k-1} - a_{k-1} = -c_{k-1} - e_{k-1} - b_{k+1}(k+1) \\ \quad - \sum_{m=k}^0 b_{k-m}(c_m - d_m + e_m - a_m) \\ -a_{k-1} + b_{k-1} = -e_{k-1} + c_{k-1} - d_{k+1}(k+1) \\ \quad + \sum_{m=k}^0 d_{k-m}(e_m - a_m + b_m - c_m). \end{cases}$$

Since $\sum_{j=0}^4 f_j = t$, it follows that

$$a_{k-1} + b_{k-1} + d_{k-1} = -c_{k-1} - e_{k-1}.$$

Therefore, if there is a rational solution of Type A (2), the coefficients a_k, b_k, d_k ($k \leq -2$) are determined inductively and it is unique.

If there exists a rational solution of Type B, we have

$$\begin{cases} f_0 = \frac{1}{3}t + a_{-1}t^{-1} + \sum_{k=-\infty}^{-2} a_k t^k, & f_1 = \frac{1}{3}t + b_{-1}t^{-1} + \sum_{k=-\infty}^{-2} b_k t^k, & f_2 = \frac{1}{3}t + c_{-1}t^{-1} + \sum_{k=-\infty}^{-2} c_k t^k, \\ f_3 = d_{-1}t^{-1} + \sum_{k=-\infty}^{-2} d_k t^k, & f_4 = e_{-1}t^{-1} + \sum_{k=-\infty}^{-2} e_k t^k, \end{cases}$$

where $a_{-1}, b_{-1}, c_{-1}, d_{-1}, e_{-1}$ have been determined in Proposition 1.2. By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we obtain

$$\begin{cases} d_{k-1} = -3(k+1)d_{k+1} + 3 \sum_{m=k}^0 d_{k-m}(e_m - a_m + b_m - c_m) \\ e_{k-1} = 3(k+1)e_{k+1} - 3 \sum_{m=k}^0 e_{k-m}(a_m - b_m + c_m - d_m). \end{cases}$$

In Proposition 1.2, we have had

$$d_{-1} = 3\alpha_3, \quad e_{-1} = -3\alpha_4.$$

Therefore the coefficients d_k, e_k ($k \leq -2$) are determined inductively and we get

$$\begin{cases} f_3 \equiv 0 \text{ if } \alpha_3 = 0, \\ f_4 \equiv 0 \text{ if } \alpha_4 = 0. \end{cases}$$

By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2, \end{cases}$$

we have

$$\begin{cases} c_{k-1} - b_{k-1} = -3(k+1)a_{k+1} + 3 \sum_{m=k}^0 a_{k-m}(b_m - c_m + d_m - e_m) \\ a_{k-1} - c_{k-1} = -3(k+1)b_{k+1} + 3 \sum_{m=k}^0 b_{k-m}(c_m - d_m + e_m - a_m) \\ b_{k-1} - a_{k-1} = -3(k+1)c_{k+1} + 3 \sum_{m=k}^0 c_{k-m}(d_m - e_m + a_m - b_m). \end{cases}$$

Since $\sum_{i=0}^4 f_i = t$, it follows that

$$a_{k-1} + b_{k-1} + c_{k-1} = d_{k-1} - e_{k-1}.$$

Therefore, if there is a rational solution of Type B, the coefficients a_k, b_k, c_k ($k \leq -2$) are determined inductively and it is unique.

If there exists a rational solution of Type C, we have

$$\begin{cases} f_0 = \frac{1}{5}t + a_{-1}t^{-1} + \sum_{k=-2}^{-\infty} a_k t^k, & f_1 = \frac{1}{5}t + b_{-1}t^{-1} + \sum_{k=-2}^{-\infty} b_k t^k, & f_2 = \frac{1}{5}t + c_{-1}t^{-1} + \sum_{k=-2}^{-\infty} c_k t^k, \\ f_3 = \frac{1}{5}t + d_{-1}t^{-1} + \sum_{k=-2}^{-\infty} d_k t^k, & f_4 = \frac{1}{5}t + e_{-1}t^{-1} + \sum_{k=-2}^{-\infty} e_k t^k, \end{cases}$$

where $a_{-1}, b_{-1}, c_{-1}, d_{-1}, e_{-1}$ have been determined in Proposition 1.2. By comparing the coefficients of the terms t^k ($k \leq -2$) in

$$\begin{cases} f'_0 = f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0 \\ f'_1 = f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1 \\ f'_2 = f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2 \\ f'_3 = f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3 \\ f'_4 = f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4, \end{cases}$$

we get

$$\begin{cases} b_{k-1} - c_{k-1} + d_{k-1} - e_{k-1} = 5(k+1)a_{k+1} - 5 \sum_{m=k}^0 a_{k-m}(b_m - c_m + d_m - e_m) \\ c_{k-1} - d_{k-1} + e_{k-1} - a_{k-1} = 5(k+1)b_{k+1} - 5 \sum_{m=k}^0 b_{k-m}(c_m - d_m + e_m - a_m) \\ d_{k-1} - e_{k-1} + a_{k-1} - b_{k-1} = 5(k+1)c_{k+1} - 5 \sum_{m=k}^0 c_{k-m}(d_m - e_m + a_m - b_m) \\ e_{k-1} - a_{k-1} + b_{k-1} - c_{k-1} = 5(k+1)d_{k+1} - 5 \sum_{m=k}^0 d_{k-m}(e_m - a_m + b_m - c_m) \\ a_{k-1} - b_{k-1} + c_{k-1} - d_{k-1} = 5(k+1)e_{k+1} - 5 \sum_{m=k}^0 e_{k-m}(a_m - b_m + c_m - d_m) \end{cases}$$

Since the rank of

$$\begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

is four, $b_{k-1}, c_{k-1}, d_{k-1}, e_{k-1}$ can be expressed by

$$a_i \ (k-1 \leq i \leq 1), \ b_j, \ c_j, \ d_j, \ e_j \ (k \leq j \leq 1).$$

Since $\sum_{k=0}^4 f_k = t$, it follows that

$$a_{k-1} + b_{k-1} + c_{k-1} + d_{k-1} + e_{k-1} = 0.$$

Therefore, if there is a rational solution of Type C, the coefficients a_k, b_k, c_k, d_k, e_k ($k \leq -2$) are determined inductively and it is unique. \square

From Proposition 1.3, we have

Corollary 1.4. *Let $(f_j)_{0 \leq j \leq 4}$ be a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$. Then, f_j ($0 \leq j \leq 4$) are odd functions.*

Proof. $A_4(\alpha_j)_{0 \leq j \leq 4}$ is invariant under the transformation

$$s_{-1} : t \longrightarrow -t, \quad f_j \longrightarrow -f_j \quad (0 \leq j \leq 4).$$

Each of Type A, Type B, Type C on Proposition 1.1 is also invariant under s_{-1} . Then $f_j(t) = -f_j(-t)$ ($0 \leq j \leq 4$), because the Laurent series of f_j at $t = \infty$ on each of types are unique. Therefore, f_j are odd functions. \square

1.2 the Laurent Series at $t = c \in \mathbb{C}$

In this subsection, we calculate the Laurent series of f_j ($0 \leq j \leq 4$) at $t = c \in \mathbb{C}$ for $A_4(\alpha_j)_{0 \leq j \leq 4}$, which are determined by Tahara [17]. The residues of f_j ($0 \leq j \leq 4$) at $t = c \in \mathbb{C}$ are integers.

Tahara [17] obtained the following proposition:

Proposition 1.5. *If some of $(f_j)_{0 \leq j \leq 4}$ have a pole at $t = c \in \mathbb{C}$, f_j is expanded as the following three types:*

(1) if f_i, f_{i+1} have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,

$$\begin{cases} f_i = (t - c)^{-1} + \frac{c}{2} + \left(1 + \frac{c^2}{12} - \frac{1}{3}\alpha_i - \frac{2}{3}\alpha_{i+1} - \frac{2}{3}\alpha_{i+3}\right)(t - c) \\ \quad + \left(-\frac{1}{2}(q_{i+2,2} + q_{i+4,2}) + \frac{c}{8} + \frac{c}{4}(\alpha_{i+2} + \alpha_{i+4})\right)(t - c)^2 + \dots \\ f_{i+1} = -(t - c)^{-1} + \frac{c}{2} + \left(1 - \frac{c^2}{12} - \frac{2}{3}\alpha_i - \frac{1}{3}\alpha_{i+1} - \frac{2}{3}\alpha_{i+3}\right)(t - c) \\ \quad + \left(-\frac{1}{2}(q_{i+2,2} + q_{i+4,2}) - \frac{1}{8}c - \frac{c}{4}(\alpha_{i+2} + \alpha_{i+4})\right)(t - c)^2 \dots \\ f_{i+2} = -\alpha_{i+2}(t - c) + q_{i+2,2}(t - c)^2 + \dots \\ f_{i+3} = \frac{\alpha_{i+3}}{3}(t - c) + 0(t - c)^2 + \dots \\ f_{i+4} = -\alpha_{i+4}(t - c) + q_{i+4,2}(t - c)^2 \dots, \end{cases}$$

where $q_{i+2,2}, q_{i+4,2}$ are arbitrary constants.

(2) if f_i, f_{i+2} have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,

$$\begin{cases} f_i = -(t - c)^{-1} + \left(\frac{1}{2}c - q_{i+3,0}\right) \\ \quad + \left(\frac{1}{3}(2 + \alpha_{i+1} - \alpha_{i+2} - 3\alpha_{i+3} - \alpha_{i+4}) + \frac{2}{3}q_{i+3,0}(c - q_{i+3,0} - 2q_{i+4,0}) - \frac{1}{3}\left(\frac{1}{2}c - q_{i+3,0}\right)^2\right) \\ \quad \times (t - c) + \dots \\ f_{i+1} = -\alpha_{i+1}(t - c) + \dots \\ f_{i+2} = (t - c)^{-1} + \left(\frac{1}{2}c - q_{i+4,0}\right) \\ \quad + \left(\frac{1}{3}(2 - \alpha_i + \alpha_{i+1} - \alpha_{i+3} - 3\alpha_{i+4}) - \frac{2}{3}q_{i+4,0}(c - 2q_{i+3,0} - q_{i+4,0}) + \frac{1}{3}\left(\frac{1}{2}c - q_{i+4,0}\right)^2\right) \\ \quad \times (t - c) + \dots \\ f_{i+3} = q_{i+3,0} + (q_{i+3,0}(-c + q_{i+3,0} + 2q_{i+4,0}) + \alpha_{i+3})(t - c) + \dots \\ f_{i+4} = q_{i+4,0} + (q_{i+4,0}(c - 2q_{i+3,0} - q_{i+4,0}) + \alpha_{i+4})(t - c) + \dots, \end{cases}$$

where $q_{i+3,0}, q_{i+4,0}$ are arbitrary constants.

(3) if $f_{i+1}, f_{i+2}, f_{i+3}, f_{i+4}$ have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,

$$\begin{cases} f_i = -\frac{\alpha_i}{3}(t - c) + \dots \\ f_{i+1} = 3(t - c)^{-1} + \left(\frac{c^2}{10} - \frac{2}{5} - \frac{3}{5}\alpha_i + \frac{3}{5}\alpha_{i+2} + \frac{1}{5}\alpha_{i+3} - \frac{1}{5}\alpha_{i+4}\right)(t - c) + \dots \\ f_{i+2} = (t - c)^{-1} + \frac{c}{2} + \left(\frac{c^2}{12} + \frac{2}{3} + \alpha_i + \frac{1}{3}\alpha_{i+1} - \frac{1}{3}\alpha_{i+3} + \frac{1}{3}\alpha_{i+4}\right)(t - c) + \dots \\ f_{i+3} = -(t - c)^{-1} + \frac{c}{2} + \left(-\frac{c^2}{12} + \frac{2}{3} + \alpha_i + \frac{1}{3}\alpha_{i+1} - \frac{1}{3}\alpha_{i+2} + \frac{1}{3}\alpha_{i+4}\right)(t - c) + \dots \\ f_{i+4} = -3(t - c)^{-1} + \left(-\frac{c^2}{10} - \frac{2}{5} - \frac{3}{5}\alpha_i - \frac{1}{5}\alpha_{i+1} + \frac{1}{5}\alpha_{i+2} + \frac{3}{5}\alpha_{i+3}\right)(t - c) + \dots. \end{cases}$$

From Proposition 1.5, we obtain the following corollary:

Corollary 1.6. Suppose that $(f_i)_{0 \leq i \leq 4}$ is a rational solution of $A_4(\alpha_i)_{0 \leq i \leq 4}$.

- (1) If $c \in \mathbb{C} \setminus \{0\}$ is a pole of f_i , $-c$ is also a pole of f_i and $\text{Res}_{t=c} f_i = \text{Res}_{t=-c} f_i$.
(2) If $\text{Res}_{t=\infty} f_i$ is an even integer, $t = 0$ is not a pole of f_i . Therefore,

$$f_i = a_{i,1}t + \sum_{j=1}^{n_i} \left(\frac{\varepsilon_{i,j}}{t - c_{i,j}} + \frac{\varepsilon_{i,j}}{t + c_{i,j}} \right),$$

where $a_{i,1} = 0, \pm 1, \frac{1}{3}, \frac{1}{5}$ and $\varepsilon_{i,j} = \pm 1, \pm 3$ and $c_{i,j} \neq 0$.

- (3) If $\text{Res}_{t=\infty} f_i$ is an odd integer, $t = 0$ is a pole of f_i . Therefore,

$$f_i = a_{i,1}t + \frac{\varepsilon_{i,0}}{t} + \sum_{j=1}^{n_i} \left(\frac{\varepsilon_{i,j}}{t - c_{i,j}} + \frac{\varepsilon_{i,j}}{t + c_{i,j}} \right),$$

where $\varepsilon_{i,0}, \varepsilon_{i,j} = \pm 1, \pm 3$ and $c_{i,j} \neq 0$.

Proof. (1) Let $c \in \mathbb{C} \setminus \{0\}$ be a pole of f_i . Then it follows from Proposition 1.5 and Corollary 1.4 that f_i has a pole at $t = c$ with the first order and is an odd function:

$$f_i(t) = -f_i(-t).$$

Therefore, $-c$ is also a pole of f_i and $\text{Res}_{t=c} f_i = \text{Res}_{t=-c} f_i$.

- (2) Suppose that $t = 0$ is a pole of f_i . Let $\pm c_1, \pm c_2, \dots, \pm c_{n_i} \in \mathbb{C} \setminus \{0\}$ be poles of f_i . Then, it follows from the residue theorem that

$$-\text{Res}_{t=\infty} f_i = \text{Res}_{t=0} f_i + 2 \sum_{j=1}^{n_i} \text{Res}_{t=c_j} f_i,$$

which is contradiction because $\text{Res}_{t=0} f_i = \pm 1$ or ± 3 .

- (3) Suppose that $t = 0$ is not a pole of f_i . Let $\pm c_1, \pm c_2, \dots, \pm c_{n_i} \in \mathbb{C} \setminus \{0\}$ be poles of f_i . Then, it follows from the residue theorem that

$$-\text{Res}_{t=\infty} f_i = 2 \sum_{j=1}^{n_i} \text{Res}_{t=c_j} f_i,$$

which is contradiction. □

2 A Necessary Condition

In this section, following Noumi and Yamada [10], we firstly introduce the shift operators of the parameters $(\alpha_i)_{0 \leq i \leq 4}$. Secondly we get a necessary condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution and prove that if $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution, α_i ($0 \leq i \leq 4$) are rational numbers. Thirdly, we transform the parameters into the set C .

Noumi and Yamada [10] defined shift operators in the following way:

Proposition 2.1. *For any $i = 0, 1, 2, 3, 4$, T_i denote shift operators which are expressed by*

$$T_1 = \pi s_4 s_3 s_2 s_1, T_2 = s_1 \pi s_4 s_3 s_2, T_3 = s_2 s_1 \pi s_4 s_3, T_4 = s_3 s_2 s_1 \pi s_4, T_0 = s_4 s_3 s_2 s_1 \pi.$$

Then,

$$T_i(\alpha_{i-1}) = \alpha_{i-1} + 1, T_i(\alpha_i) = \alpha_i - 1, T_i(\alpha_j) = \alpha_j \ (j \neq i-1, i).$$

In Proposition 1.2 and 1.5, we have determined the residues of f_i ($0 \leq i \leq 4$) at $t = \infty$, $c \in \mathbb{C}$, respectively. Therefore, the residue theorem gives a necessary condition for $A(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution.

Theorem 2.2. *If the $A_4(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution, $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfy one of the following conditions:*

- (1) *if $A_4(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution of Type A, $\alpha_i \in \mathbb{Z}$ ($0 \leq i \leq 4$);*
- (2) *if $A_4(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution of Type B, for some $i = 0, 1, 2, 3, 4$,*

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \left(\frac{n_1}{3} - \frac{n_3}{3}, \frac{n_1}{3}, \frac{n_1}{3} + \frac{n_4}{3}, \frac{n_3}{3}, -\frac{n_4}{3} \right) \pmod{\mathbb{Z}},$$

where $n_1, n_3, n_4 = 0, 1, 2$;

- (3) *if $A_4(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution of Type C, for some $i = 0, 1, 2, 3, 4$,*

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \left(\frac{n_1}{5} + \frac{2n_2}{5} + \frac{3n_3}{5}, \frac{n_1}{5} + \frac{2n_2}{5} + \frac{n_3}{5}, \frac{n_1}{5}, \frac{n_1}{5} + \frac{n_2}{5}, \frac{n_1}{5} + \frac{n_3}{5} \right) \pmod{\mathbb{Z}},$$

where $n_1, n_2, n_3 = 0, 1, 2, 3, 4$.

In (1), (2) and (3), we consider the suffix of the parameters α_i as elements of $\mathbb{Z}/5\mathbb{Z}$.

Proof. Proposition 1.5 implies that $\text{Res}_{t=c} f_i = \pm 1, \pm 3$ ($0 \leq i \leq 4$) for $t = c \in \mathbb{C}$. Therefore, it follows from the residue theorem that $\text{Res}_{t=\infty} f_i \in \mathbb{Z}$ ($0 \leq i \leq 4$).

If Type A (1) occurs, it follows from Proposition 1.2 that $\alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4} \in \mathbb{Z}$, which proves that $\alpha_i \in \mathbb{Z}$ because $\sum_{k=0}^4 \alpha_k = 1$.

If Type A (2) occurs, we can show that $\alpha_j \in \mathbb{Z}$ ($0 \leq j \leq 4$) in the same way as Type A (1).

If Type B occurs, it follows from Proposition 1.2 that $\text{Res}_{t=\infty} f_{i+3}$ and $\text{Res}_{t=\infty} f_{i+4} \in \mathbb{Z}$, which means that

$$\alpha_{i+3} = \frac{n_3}{3}, \alpha_{i+4} = -\frac{n_4}{3}, n_3, n_4 \in \mathbb{Z}.$$

Furthermore, Proposition 1.2 implies that $\text{Res}_{t=\infty} f_{i+1}$ and $\text{Res}_{t=\infty} f_{i+2} \in \mathbb{Z}$, which shows that

$$\begin{aligned} \alpha_{i+2} - \alpha_i - \frac{n_3}{3} - \frac{n_4}{3} &= m_1 \in \mathbb{Z} \\ \alpha_i - \alpha_{i+1} + \frac{n_3}{3} - n_4 &= m_2 \in \mathbb{Z}. \end{aligned}$$

By solving this system of equations of α_i, α_{i+2} , we obtain

$$\begin{aligned} \alpha_i &= \alpha_{i+1} - \frac{n_3}{3} + m_2 + n_4 \\ \alpha_{i+1} &= \alpha_{i+1} \\ \alpha_{i+2} &= \alpha_{i+1} + \frac{n_4}{3} + m_1 + m_2 + n_4. \end{aligned}$$

Since $\alpha_{i+3} = \frac{n_3}{3}, \alpha_{i+4} = -\frac{n_4}{3}$ and $\sum_{j=0}^4 \alpha_j = 1$, it follows that $\alpha_{i+1} = \frac{n_1}{3}$ for some integer $n_1 \in \mathbb{Z}$, which implies that

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \left(\frac{n_1}{3} - \frac{n_3}{3}, \frac{n_1}{3}, \frac{n_1}{3} + \frac{n_4}{3}, \frac{n_3}{3}, -\frac{n_4}{3} \right) \pmod{\mathbb{Z}}.$$

If Type C occurs, it follows from Proposition 1.2 that

$$\begin{aligned} 3\alpha_1 + \alpha_2 - \alpha_3 - 3\alpha_4 &= m_0 \in \mathbb{Z} \\ 3\alpha_2 + \alpha_3 - \alpha_4 - 3\alpha_0 &= m_1 \in \mathbb{Z} \\ 3\alpha_3 + \alpha_4 - \alpha_0 - 3\alpha_1 &= m_2 \in \mathbb{Z} \\ 3\alpha_4 + \alpha_0 - \alpha_1 - 3\alpha_2 &= m_3 \in \mathbb{Z} \\ 3\alpha_0 + \alpha_1 - \alpha_2 - 3\alpha_3 &= m_4 \in \mathbb{Z}. \end{aligned}$$

By solving this system of equations, we obtain

$$\begin{aligned}
\alpha_0 &= \alpha_3 - \frac{3}{5}m_0 - \frac{2}{5}m_1 - \frac{2}{5}m_2 - \frac{1}{5}m_3 \\
\alpha_1 &= \alpha_3 + \frac{1}{5}m_1 - \frac{2}{5}m_2 + \frac{1}{5}m_3 \\
\alpha_2 &= \alpha_3 - \frac{4}{5}m_0 - \frac{3}{5}m_2 - \frac{3}{5}m_3 \\
\alpha_3 &= \alpha_3 \\
\alpha_4 &= \alpha_3 - \frac{3}{5}m_0 + \frac{1}{5}m_1 - \frac{3}{5}m_2.
\end{aligned}$$

Since $\sum_{i=0}^4 \alpha_i = 1$, it follows that

$$\alpha_j = \frac{n_j}{5} \quad n_j \in \mathbb{Z} \quad (0 \leq j \leq 4).$$

We substitute $\alpha_j = \frac{n_j}{5}$ into the residues of f_j at $t = \infty$ again and get

$$\begin{aligned}
3n_1 + n_2 - n_3 - 3n_4 &\equiv 0 \pmod{5} \\
3n_2 + n_3 - n_4 - 3n_0 &\equiv 0 \pmod{5} \\
3n_3 + n_4 - n_0 - 3n_1 &\equiv 0 \pmod{5} \\
3n_4 + n_0 - n_1 - 3n_2 &\equiv 0 \pmod{5} \\
3n_0 + n_1 - n_2 - 3n_3 &\equiv 0 \pmod{5}.
\end{aligned}$$

By solving this system of equations in the field $\mathbb{Z}/5\mathbb{Z}$, we obtain

$$\begin{aligned}
n_0 &\equiv l_1 + 2l_2 + 3l_3 \pmod{5} \\
n_1 &\equiv l_1 + 2l_2 + l_3 \pmod{5} \\
n_2 &\equiv l_1 \pmod{5} \\
n_3 &\equiv l_1 + l_2 \pmod{5} \\
n_4 &\equiv l_1 + l_3 \pmod{5}.
\end{aligned}$$

□

By the Bäcklund transformations, we can transform the parameters obtained in Theorem 2.2 into the set C . For the purpose, we study the relationship between the Bäcklund transformations s_i ($0 \leq i \leq 4$) and Type A, Type B, Type C on Proposition 1.1 in the following proposition:

Proposition 2.3. *The Bäcklund transformation s_i preserves Type A, Type B and Type C on Proposition 1.1.*

Type A (1): for some $j = 0, 1, 2, 3, 4$, f_j has a pole at $t = \infty$. When $j = i, i \pm 1$, s_i preserves Type A (1). When $j = i \pm 2$, s_i changes Type A (1) into Type A (2).

Type A (2): for some $j = 0, 1, 2, 3, 4$, f_j, f_{j+1}, f_{j+3} have a pole at $t = \infty$. When $j = i, i - 1, i + 2$, s_i preserves Type A (2). When $j = i + 1, i - 2$, s_i changes Type A (2) into Type A (1).

Type B and C are invariant under the Bäcklund transformations.

With the Bäcklund transformations, we transform the parameters $(\alpha_i)_{0 \leq i \leq 4}$ in Theorem 2.2 into the set C . In the set C , we have one, five, six kinds of parameters which correspond to the parameters in (1), (2), (3) in Theorem 2.2, respectively.

Theorem 2.4. *By some Bäcklund transformations, the parameters in (1), (2), (3) in Theorem 2.2 can be transformed into the following parameters in the set C , respectively.*

(1) *The parameters are transformed into $(1, 0, 0, 0, 0)$.*

(2) *The parameters in Theorem 2.2 (2) are transformed into one of*

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right), \left(\frac{2}{3}, 0, 0, \frac{1}{3}, 0\right), \left(\frac{1}{3}, 0, 0, \frac{2}{3}, 0\right), \left(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right), (1, 0, 0, 0, 0).$$

The parameters in Theorem 2.2 (2) are transformed into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ if and only if

$$(n_1, n_3, n_4) = (\pm 1, 0, 0), (\pm 1, 0, \pm 1), (\pm 1, \pm 1, 0), \pm(0, 1, -1),$$

or if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \pm \frac{1}{3}(1, 1, 1, 0, 0) & \text{mod } \mathbb{Z} \\ \pm \frac{1}{3}(1, -1, -1, 1, 0) & \text{mod } \mathbb{Z}. \end{cases}$$

(3) *The parameters in Theorem 2.2 (3) are transformed into one of*

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), (1, 0, 0, 0, 0), \left(\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0\right), \left(\frac{1}{5}, 0, \frac{2}{5}, \frac{2}{5}, 0\right), \left(\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}\right), \left(\frac{3}{5}, \frac{1}{5}, 0, 0, \frac{1}{5}\right).$$

The parameters in Theorem 2.2 (3) are transformed into $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ if and only if

$$\begin{aligned}(n_1, n_2, n_3) &= (1, 0, 0), (1, 2, 2), (1, 0, 1), (1, 2, 3), (1, 1, 0), (2, 0, 0), \\ &= (2, 4, 4), (2, 0, 2), (2, 4, 1), (2, 2, 0), (2, 2, 1), (3, 0, 0), \\ &= (3, 1, 1), (3, 3, 4), (3, 3, 0), (3, 0, 3), (3, 1, 4), (4, 0, 0), \\ &= (4, 3, 3), (4, 4, 0), (4, 4, 2), (4, 3, 2), (4, 0, 4),\end{aligned}$$

or if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \frac{j}{5}(1, 1, 1, 1, 1) & \text{mod } \mathbb{Z} \\ \frac{j}{5}(1, 2, 1, 3, 3) & \text{mod } \mathbb{Z}, \end{cases}$$

with some $j = 1, 2, 3, 4$.

Proof. (1) We inductively prove that the parameters $(n_0, n_1, n_2, n_3, n_4)$ $n_i \in \mathbb{Z}$ can be transformed into $(1, 0, 0, 0, 0)$.

i) Four of the parameters are 0.

By π , the parameters can be transformed into $(1, 0, 0, 0, 0)$.

ii) Three of the parameters are 0.

(1) By $T_1^{n_1}$, we have $(n_0, n_1, 0, 0, 0) \longrightarrow (n_0, 0, 0, 0, 0)$,

(2) By $T_2^{n_2}$, we get $(n_0, 0, n_2, 0, 0) \longrightarrow (n_0, n_2, 0, 0, 0)$,

iii) Two of the parameters are 0.

(1) By $T_2^{n_2}$, we obtain $(n_0, n_1, n_2, 0, 0) \longrightarrow (n_0, n_1 + n_2, 0, 0, 0)$,

(2) By $T_3^{n_3}$, we have $(n_0, n_1, 0, n_3, 0) \longrightarrow (n_0, n_1, n_3, 0, 0, 0)$,

iv) One of the parameters is 0.

By $T_3^{n_3}$, we get $(n_0, n_1, n_2, n_3, 0) \longrightarrow (n_0, n_1, n_2 + n_3, 0, 0)$.

v) None of the parameters is 0.

By $T_4^{n_4}$, we obtain $(n_0, n_1, n_2, n_3, n_4) \longrightarrow (n_0, n_1, n_2, n_3 + n_4, 0)$,

(2) By some Bäcklund transformations, we can transform the parameters

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{n_1}{3} - \frac{n_3}{3}, \frac{n_1}{3}, \frac{n_1}{3} + \frac{n_4}{3}, \frac{n_3}{3}, -\frac{n_4}{3} \right) \text{ mod } \mathbb{Z}, \quad n_1, n_3, n_4 = 0, 1, 2,$$

into the set C . We have to consider $3^3 = 27$ cases. Here, we show that $(\alpha_i)_{0 \leq i \leq 4}$ can be transformed into the set C in the following five cases. The other cases can be proved in the same way.

When $n_1 = n_3 = n_4 = 0$, the discussion on (1) implies that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longrightarrow (1, 0, 0, 0, 0).$$

When $n_1 = 1, n_3 = 0, n_4 = 2$, by π , we get

$$(\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}) \longrightarrow (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0).$$

When $n_1 = 1 = n_3 = n_4 = 1$, by $s_0 \circ s_4$, we obtain

$$(0, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, -\frac{1}{3}) \longrightarrow (\frac{1}{3}, 0, \frac{2}{3}, 0, 0).$$

When $n_1 = 1, n_3 = n_4 = 2$, by s_0 , we have

$$(-\frac{1}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}) \longrightarrow (\frac{1}{3}, 0, 0, \frac{2}{3}, 0).$$

When $n_1 = 1 = n_3 = 1, n_4 = 2$, by π , we get

$$(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) \longrightarrow (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0).$$

(3) By some Bäcklund transformations, we can transform the parameters

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{n_1}{5} + \frac{2n_2}{5} + \frac{3n_3}{5}, \frac{n_1}{5} + \frac{2n_2}{5} + \frac{n_3}{5}, \frac{n_1}{5}, \frac{n_1}{5} + \frac{n_2}{5}, \frac{n_1}{5} + \frac{n_3}{5}) \bmod \mathbb{Z}$$

$$n_1, n_2, n_3 = 0, 1, 2, 3, 4$$

into the set C . We have to consider $5^3 = 125$ cases. Here, we only prove that $(\alpha_i)_{0 \leq i \leq 4}$ can be transformed into the set C in the following six cases. The other cases can be proved in the same way.

When $n_1 = n_2 = n_3 = 0$, by some shift operators, we get

$$(\alpha_i)_{0 \leq i \leq 4} \longrightarrow (1, 0, 0, 0, 0).$$

When $n_1 = 1, n_2 = n_3 = 0$, by some shift operators, we obtain

$$(\alpha_i)_{0 \leq i \leq 4} \longrightarrow (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}).$$

When $n_1 = 0, n_2 = 2, n_3 = 0$, by $\pi^{-1} \circ s_4 \circ s_0$, we have

$$(-\frac{1}{5}, \frac{4}{5}, 0, \frac{2}{5}, 0) \longrightarrow (\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0).$$

When $n_1 = 0, n_2 = 1, n_3 = 0$, by some shift operators, we get

$$(\alpha_i)_{0 \leq i \leq 4} \longrightarrow \left(\frac{2}{5}, \frac{2}{5}, 0, \frac{1}{5}, 0\right).$$

When $n_1 = n_2 = 0, n_3 = 2$, by some shift operators, we obtain

$$(\alpha_i)_{0 \leq i \leq 4} \longrightarrow \left(\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}\right).$$

When $n_1 = n_2 = 0, n_3 = 1$, by some shift operators, we have

$$(\alpha_i)_{0 \leq i \leq 4} \longrightarrow \left(\frac{3}{5}, \frac{1}{5}, 0, 0, \frac{1}{5}\right).$$

□

3 A Sufficient Condition

In the previous section, we have shown a necessary condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution and have transformed $(\alpha_i)_{0 \leq i \leq 4} \in \mathbb{R}^5$ into the set C .

In this section, following Noumi and Yamada [10], we firstly introduce the Hamiltonian H for $A_4(\alpha_i)_{0 \leq i \leq 4}$ and its principal part \hat{H} . Secondly, from Proposition 1.2 and 1.5, we calculate the residues of \hat{H} at $t = \infty, c$. Thirdly, by the residue calculus of \hat{H} , we decide a sufficient condition for $A_4(\alpha_i)_{0 \leq i \leq 4}$ to have a rational solution.

Noumi and Yamada [11] defined the Hamiltonian H of $A_4(\alpha_j)_{0 \leq j \leq 4}$ by

$$\begin{aligned} H &= f_0 f_1 f_2 + f_1 f_2 f_3 + f_2 f_3 f_4 + f_3 f_4 f_0 + f_4 f_0 f_1 \\ &+ \frac{1}{5} (2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4) f_0 + \frac{1}{5} (2\alpha_1 + 4\alpha_2 + \alpha_3 + 3\alpha_4) f_1 \\ &- \frac{1}{5} (3\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4) f_2 + \frac{1}{5} (2\alpha_1 - \alpha_2 + \alpha_3 + 3\alpha_4) f_3 \\ &- \frac{1}{5} (3\alpha_1 + \alpha_2 + 4\alpha_3 + 2\alpha_4) f_4. \end{aligned}$$

\hat{H} denotes the principal part of H which is defined by the equation

$$\hat{H} = f_0 f_1 f_2 + f_1 f_2 f_3 + f_2 f_3 f_4 + f_3 f_4 f_0 + f_4 f_0 f_1.$$

We suppose that $(f_j)_{0 \leq j \leq 4}$ is a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$. The order of a pole of \hat{H} at $t = \infty$ is at most three, because Proposition 1.1 implies that f_i ($0 \leq i \leq 4$) have

a pole at $t = \infty$ with the first order or are regular at $t = \infty$. Since Corollary 1.4 shows that f_i ($0 \leq i \leq 4$) are odd functions, the Laurent series of \hat{H} at $t = \infty$ are given by

$$\hat{H} := h_{\infty,3}t^3 + h_{\infty,1}t + h_{\infty,-1}t^{-1} + O(t^{-3}) \text{ at } t = \infty.$$

In the following lemma, we calculate $h_{\infty,-1}$ by using the Laurent series of f_i ($0 \leq i \leq 4$) at $t = \infty$ in Proposition 1.2.

Lemma 3.1. *Suppose that $(f_j)_{0 \leq j \leq 4}$ is a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$.*

Type A (1): for some $i = 0, 1, 2, 3, 4$, f_i has a pole at $t = \infty$. Then,

$$h_{\infty,-1} = -\alpha_{i+1}\alpha_{i+2} - \alpha_{i+3}\alpha_{i+4} - \alpha_{i+4}\alpha_{i+1}.$$

Type A (2): for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$. Then,

$$h_{\infty,-1} = -\alpha_{i+2}(\alpha_i + \alpha_{i+3}) + \alpha_{i+4}(\alpha_{i+1} + \alpha_{i+3}) + \alpha_{i+2}\alpha_{i+4}.$$

Type B: for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+2} have a pole at $t = \infty$. Then,

$$h_{\infty,-1} = \frac{1}{3} \left\{ -(\alpha_i - \alpha_{i+1} + \alpha_{i+3})^2 - (\alpha_{i+2} - \alpha_i - \alpha_{i+3} + \alpha_{i+4})(\alpha_{i+2} + \alpha_{i+4} - \alpha_{i+1}) - 9\alpha_{i+3}\alpha_{i+4} \right\}.$$

Type C: f_0, f_1, f_2, f_3, f_4 have a pole at $t = \infty$. Then,

$$h_{\infty,-1} = \frac{1}{5}(-a_{-1}^2 + a_{-1}e_{-1} - b_{-1}^2 - a_{-1}c_{-1} - c_{-1}^2 + c_{-1}d_{-1} + 2d_{-1}e_{-1}),$$

where

$$\begin{aligned} a_{-1} &= 3\alpha_1 + \alpha_2 - \alpha_3 - 3\alpha_4, b_{-1} = 3\alpha_2 + \alpha_3 - \alpha_4 - 3\alpha_0, c_{-1} = 3\alpha_3 + \alpha_4 - \alpha_0 - 3\alpha_1, \\ d_{-1} &= 3\alpha_4 + \alpha_0 - \alpha_1 - 3\alpha_2, e_{-1} = 3\alpha_0 + \alpha_1 - \alpha_2 - 3\alpha_3. \end{aligned}$$

In the following lemma, we decide the residue of \hat{H} at $t = c$ by using the Laurent series of f_i ($0 \leq i \leq 4$) in Proposition 1.5.

Lemma 3.2. *Suppose that $(f_i)_{0 \leq i \leq 4}$ is a rational solution of $A_4(\alpha_i)_{0 \leq i \leq 4}$ and some rational functions of $(f_i)_{0 \leq i \leq 4}$ have a pole at $t = c \in \mathbb{C}$. Then the residue of \hat{H} at $t = c$ is as follows:*

(1) *if f_i, f_{i+1} have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,*

$$\text{Res}_{t=c} \hat{H} = \alpha_{i+2} + \alpha_{i+4};$$

(2) if f_i, f_{i+2} have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,

$$\text{Res}_{t=c} \hat{H} = \alpha_{i+1};$$

(3) if $f_{i+1}, f_{i+2}, f_{i+3}, f_{i+4}$ have a pole at $t = c \in \mathbb{C}$ for some $i = 0, 1, 2, 3, 4$,

$$\text{Res}_{t=c} \hat{H} = \alpha_{i+1} + \alpha_{i+4}.$$

From now on, let us study a rational solution of $A_4(\alpha_j)_{0 \leq j \leq 4}$ when $(\alpha_i)_{0 \leq i \leq 4}$ is in the set C . For the purpose, we have the following lemma:

Lemma 3.3. Suppose that the parameters $(\alpha_j)_{0 \leq j \leq 4} \in \mathbb{R}^5$ are in the set C . If $A_4(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution $(f_j)_{0 \leq j \leq 4}$, then,

$$h_{\infty, -1} \geq 0.$$

Proof. Let $c_1, \dots, c_k \in \mathbb{C}$ be the poles of $(f_j)_{0 \leq j \leq 4}$. Since $0 \leq \alpha_i \leq 1$ ($0 \leq i \leq 4$), it follows from Lemma 3.2 that

$$\text{Res}_{t=c_l} \hat{H} \geq 0 \quad (1 \leq l \leq k).$$

Therefore it follows from the residue theorem that

$$h_{\infty, -1} = -\text{Res}_{t=\infty} \hat{H} = \sum_{l=1}^k \text{Res}_{t=c_l} \hat{H} \geq 0.$$

□

For the residue calculus of \hat{H} , we make two tables about the residues of \hat{H} at $t = c \in \mathbb{C}$.

Table 1: the residues of \hat{H} at $t = c \in \mathbb{C}$ in the case of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$

i	0	1	2	3	4
$\alpha_{i+2} + \alpha_{i+4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
α_{i+1}	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
$\alpha_{i+1} + \alpha_{i+4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

By using Table 2, we study a rational solution of Type A of $A_4(1, 0, 0, 0, 0)$.

Table 2: the residues of \hat{H} at $t = c \in \mathbb{C}$ in the case of $(1, 0, 0, 0, 0)$

i	0	1	2	3	4
$\alpha_{i+2} + \alpha_{i+4}$	0	1	0	1	0
α_{i+1}	0	0	0	0	1
$\alpha_{i+1} + \alpha_{i+4}$	0	1	0	0	1

Lemma 3.4. $A_4(1, 0, 0, 0, 0)$ has a unique rational solution of Type A which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0).$$

Proof. If $A_4(1, 0, 0, 0, 0)$ has a rational solution of Type A, it follows from Lemma 3.1 that $h_{\infty, -1} = 0$. Furthermore Lemma 3.2 and Table 2 imply that the residue of \hat{H} at $t = c \in \mathbb{C}$ is nonnegative. Then it follows from the residue theorem that $\text{Res}_{t=c} \hat{H} = 0$. Therefore, Table 2 implies that

$$\begin{aligned} & (f_0, f_1), (f_2, f_3), (f_4, f_0) \\ & (f_0, f_2), (f_1, f_3), (f_2, f_4), (f_3, f_0) \\ & (f_1, f_2, f_3, f_4), (f_3, f_4, f_0, f_1), (f_4, f_0, f_1, f_2) \end{aligned}$$

can have a pole at $t = c \in \mathbb{C}$.

Proposition 1.1 shows that Type A (1) and Type A (2) can occur.

Type A (1): for some $i = 0, 1, 2, 3, 4$, f_i has a pole at $t = \infty$. If f_0 has a pole at $t = \infty$, it follows from the uniqueness in Proposition 1.3 that

$$(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0).$$

We suppose that f_1 has a pole at $t = \infty$ and show contradiction. The other four cases can be proved in the same way. Proposition 1.2 implies that

$$-\text{Res}_{t=\infty} f_1 = 1, f_2 = f_3 = f_4 \equiv 0, -\text{Res}_{t=\infty} f_0 = -1.$$

Since $f_2 = f_3 = f_4 \equiv 0$, only (f_0, f_1) can have a pole in \mathbb{C} . It follows from Proposition 1.5 that $\text{Res}_{t=0} f_0 = 1$, $\text{Res}_{t=0} f_1 = -1$, which contradicts the residue theorem.

Type A (2): for some $i = 0, 1, 2, 3, 4$, f_i, f_{i+1}, f_{i+3} have a pole at $t = \infty$.

When f_0, f_1, f_3 have a pole at $t = \infty$, Proposition 1.2 shows that

$$-\text{Res}_{t=\infty} f_0 = 0, -\text{Res}_{t=\infty} f_1 = 1, f_2 \equiv 0, -\text{Res}_{t=\infty} f_3 = -1, f_4 \equiv 0.$$

Since $f_2 = f_4 \equiv 0$,

$$(f_0, f_1), (f_1, f_3), (f_3, f_0)$$

can have a pole in \mathbb{C} . If (f_0, f_1) or (f_3, f_0) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_0 = 1$, which contradicts the residue theorem. If (f_1, f_3) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_1 = -1$, $\text{Res}_{t=c}f_3 = 1$, which contradicts the residue theorem.

When f_1, f_2, f_4 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty}f_1 = 1, \quad -\text{Res}_{t=\infty}f_2 = 3, \quad f_3 \equiv 0, \quad -\text{Res}_{t=\infty}f_4 = -3, \quad -\text{Res}_{t=\infty}f_0 = -1.$$

Therefore

$$(f_0, f_1), (f_4, f_0) (f_0, f_2), (f_2, f_4), (f_4, f_0, f_1, f_2)$$

can have a pole in \mathbb{C} because $f_3 \equiv 0$. When (f_4, f_0) , (f_2, f_4) , (f_4, f_0, f_1, f_2) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_4 = 1, 3$, which contradicts the residue theorem. If (f_0, f_1) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_1 = -1$, which contradicts the residue theorem. If (f_0, f_2) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_0 = -1$, $\text{Res}_{t=c}f_2 = 1$, which contradicts the residue theorem.

When f_2, f_3, f_0 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty}f_2 = 1, \quad -\text{Res}_{t=\infty}f_3 = 1, \quad f_4 \equiv 0, \quad -\text{Res}_{t=\infty}f_0 = -1, \quad -\text{Res}_{t=\infty}f_1 = -1.$$

Then Corollary 1.6 shows that (f_0, f_1, f_2, f_3) have a pole at $t = 0$ because $\text{Res}_{t=\infty}f_j$ ($0 \leq j \leq 4$) are odd integers. Lemma 3.2 implies that $\text{Res}_{t=0}\hat{H} = 1$. Since $-\text{Res}_{t=\infty}\hat{H} = h_{\infty,-1} = 0$ and $\text{Res}_{t=c}\hat{H}$ is nonnegative, this contradicts the residue theorem.

When f_3, f_4, f_1 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty}f_3 = 1, \quad -\text{Res}_{t=\infty}f_4 = -1, \quad -\text{Res}_{t=\infty}f_0 = 1, \quad -\text{Res}_{t=\infty}f_1 = -1, \quad f_2 \equiv 0.$$

Therefore

$$(f_0, f_1), (f_4, f_0) (f_1, f_3), (f_3, f_0) (f_3, f_4, f_0, f_1)$$

can have a pole in \mathbb{C} . Since $f_4 \not\equiv 0$ and $\text{Res}_{t=\infty}f_4 \neq 0$, f_4 has a pole at $t = c \in \mathbb{C}$. If (f_4, f_0) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_4 = 1$, which contradicts the residue theorem. If (f_3, f_4, f_0, f_1) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_4 = 3$, which contradicts the residue theorem.

When f_4, f_0, f_2 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty}f_4 = 1, \quad -\text{Res}_{t=\infty}f_0 = 0, \quad f_1 \equiv 0, \quad -\text{Res}_{t=\infty}f_2 = -1, \quad f_3 \equiv 0.$$

Since $f_1 = f_3 \equiv 0$,

$$(f_4, f_0) (f_0, f_2), (f_2, f_4)$$

can have a pole in \mathbb{C} . When (f_4, f_0) or (f_0, f_2) have a pole at $t = c \in \mathbb{C}$, Proposition 1.5 shows that $\text{Res}_{t=c} f_0 = -1$, which contradicts the residue theorem. Therefore, f_0 is regular in \mathbb{C} and (f_2, f_4) have a pole at $t = c$ because $\text{Res}_{t=\infty} f_2$ and $\text{Res}_{t=\infty} f_4$ are not zero. Proposition 1.5 and Corollary 1.6 imply that

$$f_4 = t + \frac{1}{t}, \quad f_0 = t, \quad f_1 \equiv 0, \quad f_2 = -t - \frac{1}{t}, \quad f_3 \equiv 0,$$

because $\text{Res}_{t=\infty} f_4$ and $\text{Res}_{t=\infty} f_2$ are odd integers. By substituting this solution into $A_4(1, 0, 0, 0, 0)$, we can show contradiction. \square

By using Table 1, we study a rational solution of Type B of $A_4(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$.

Lemma 3.5. $A_4(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ has a unique rational solution of Type B which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0).$$

Proof. Proposition 1.2 implies that f_i, f_{i+1}, f_{i+2} can have a pole at $t = \infty$ for some $i = 0, 1, 2, 3, 4$.

If f_0, f_1, f_2 have a pole at $t = \infty$, Proposition 1.2 and 1.3 show that

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0).$$

If f_1, f_2, f_3 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty} f_1 = -\text{Res}_{t=\infty} f_2 = 0, \quad -\text{Res}_{t=\infty} f_3 = 1, \quad f_4 \equiv 0, \quad -\text{Res}_{t=\infty} f_0 = -1.$$

Lemma 3.1 shows that $h_{\infty, -1} = 0$. Furthermore Lemma 3.2 and Table 1 implies that $\text{Res}_{t=c} \hat{H}$ is nonnegative when $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. Thus, the residue theorem shows that $\text{Res}_{t=c} \hat{H} = 0$. Therefore Lemma 3.2 and Table 1 implies that only (f_2, f_4) and (f_3, f_0) can have a pole at $t = c \in \mathbb{C}$. Since $f_4 \equiv 0$, f_4 cannot have a pole in \mathbb{C} . If (f_3, f_0) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c} f_3 = -1$ and $\text{Res}_{t=c} f_0 = 1$, which contradicts the residue theorem.

If f_2, f_3, f_4 have a pole at $t = \infty$, it follows from Lemma 3.1 that $h_{\infty, -1} = -\frac{4}{9}$, which contradicts Lemma 3.3.

If f_3, f_4, f_0 have a pole at $t = \infty$, it follows from Lemma 3.1 that $h_{\infty, -1} = -\frac{10}{27}$, which contradicts Lemma 3.3.

If f_4, f_0, f_1 have a pole at $t = \infty$, it follows from Proposition 1.2 that

$$-\text{Res}_{t=\infty} f_4 = -1 - \text{Res}_{t=\infty} f_0 = 0, \quad -\text{Res}_{t=\infty} f_1 = 0, \quad -\text{Res}_{t=\infty} f_2 = 1, \quad f_3 \equiv 0.$$

Lemma 3.1 implies that $h_{\infty,-1} = 0$. Table 1 shows that $\text{Res}_{t=c}\hat{H}$ is nonnegative when $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. Thus, it follows from the residue theorem that $\text{Res}_{t=c}\hat{H} = 0$ for any $c \in \mathbb{C}$. Therefore, Lemma 3.2 and Table 1 imply that only (f_2, f_4) and (f_3, f_0) can have a pole in \mathbb{C} . Since $f_3 \equiv 0$, f_3 cannot have a pole in \mathbb{C} . If (f_2, f_4) have a pole at $t = c \in \mathbb{C}$, it follows from Proposition 1.5 that $\text{Res}_{t=c}f_2 = -1$ and $\text{Res}_{t=c}f_4 = 1$, which contradicts the residue theorem. \square

By using Lemma 3.3, we prove the following lemma:

Lemma 3.6. $A_4(\frac{2}{3}, 0, 0, \frac{1}{3}, 0), A_4(\frac{1}{3}, 0, 0, \frac{2}{3}, 0), A_4(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}), A_4(1, 0, 0, 0, 0)$ do not have a rational solution of Type B.

Proof. If the equations in the lemma have a rational solution of Type B, it follows from Lemma 3.1 that $h_{\infty,-1} < 0$, which contradicts Lemma 3.3. \square

From Proposition 1.1, 1.2 and 1.3, we prove the following lemma:

Lemma 3.7. $A_4(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ has a unique rational solution of Type C which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}).$$

By using Lemma 3.3, we prove the following lemma:

Lemma 3.8. $A_4(1, 0, 0, 0, 0), A_4(\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0), A_4(\frac{1}{5}, 0, \frac{2}{5}, \frac{2}{5}, 0), A_4(\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}), A_4(\frac{3}{5}, \frac{1}{5}, 0, 0, \frac{1}{5})$ do not have a rational solution of Type C.

Proof. If the equations in the lemma have a rational solution of Type C, it follows from Lemma 3.1 that $h_{\infty,-1} < 0$, which contradicts Lemma 3.3. \square

Theorem 2.2 proves that if $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution of Type A, the parameters α_i ($0 \leq i \leq 4$) are integers. Theorem 2.4 shows that $(\alpha_i)_{0 \leq i \leq 4}$ can be transformed into $(1, 0, 0, 0, 0)$. Lemma 3.4 proves that $A_4(1, 0, 0, 0, 0)$ has a unique rational solution of Type A which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0).$$

Therefore, $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution of Type A if and only if α_i ($0 \leq i \leq 4$) are integers. Furthermore, the rational solution is unique and can be transformed into

$$(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0, 0).$$

Theorem 2.2 implies that if $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution of Type B, for some $i = 0, 1, 2, 3, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv (\frac{n_1}{3} - \frac{n_3}{3}, \frac{n_1}{3}, \frac{n_1}{3} + \frac{n_4}{3}, \frac{n_3}{3}, -\frac{n_4}{3}) \pmod{\mathbb{Z}},$$

where $n_1, n_3, n_4 = 0, 1, 2$. Theorem 2.4 shows that the parameters $(\alpha_i)_{0 \leq i \leq 4}$ can be transformed into one of

$$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0), (\frac{2}{3}, 0, 0, \frac{1}{3}, 0), (\frac{1}{3}, 0, 0, \frac{2}{3}, 0), (0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}), (1, 0, 0, 0, 0)$$

and that the parameters $(\alpha_i)_{0 \leq i \leq 4}$ are transformed into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \pm \frac{1}{3}(1, 1, 1, 0, 0) & \text{mod } \mathbb{Z} \\ \pm \frac{1}{3}(1, -1, -1, 1, 0) & \text{mod } \mathbb{Z}. \end{cases}$$

Lemma 3.5 shows that $A_4(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ has a unique rational solution which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0).$$

Lemma 3.6 shows that $A_4(\frac{2}{3}, 0, 0, \frac{1}{3}, 0), A_4(\frac{1}{3}, 0, 0, \frac{2}{3}, 0), A_4(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}), A_4(1, 0, 0, 0, 0)$ do not have a rational solution of Type B. Therefore, $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution of Type B if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \pm \frac{1}{3}(1, 1, 1, 0, 0) & \text{mod } \mathbb{Z} \\ \pm \frac{1}{3}(1, -1, -1, 1, 0) & \text{mod } \mathbb{Z}. \end{cases}$$

Furthermore the rational solution is unique and can be transformed into

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0).$$

Theorem 2.2 proves that if $A_4(\alpha_k)_{0 \leq k \leq 4}$ has a rational solution of Type C, for some $i = 0, 1, 2, 3, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv (\frac{n_1}{5} + \frac{2n_2}{5} + \frac{3n_3}{5}, \frac{n_1}{5} + \frac{2n_2}{5} + \frac{n_3}{5}, \frac{n_1}{5}, \frac{n_1}{5} + \frac{n_2}{5}, \frac{n_1}{5} + \frac{n_3}{5}) \text{ mod } \mathbb{Z},$$

where $n_1, n_2, n_3 = 0, 1, 2, 3, 4$. Theorem 2.4 shows that the parameters $(\alpha_k)_{0 \leq k \leq 4}$ can be transformed into one of

$$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), (1, 0, 0, 0, 0), (\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0), (\frac{1}{5}, 0, \frac{2}{5}, \frac{2}{5}, 0), (\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5}), (\frac{3}{5}, \frac{1}{5}, 0, 0, \frac{1}{5})$$

and that the parameters $(\alpha_k)_{0 \leq k \leq 4}$ are transformed into $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \frac{i}{5}(1, 1, 1, 1, 1) & \text{mod } \mathbb{Z} \\ \frac{i}{5}(1, 2, 1, 3, 3) & \text{mod } \mathbb{Z}, \end{cases}$$

with some $j = 1, 2, 3, 4$. Lemma 3.7 implies that $A_4(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ has a unique rational solution of Type C which is given by

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}).$$

Lemma 3.8 shows that $A_4(1, 0, 0, 0, 0)$, $A_4(\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, 0)$, $A_4(\frac{1}{5}, 0, \frac{2}{5}, \frac{2}{5}, 0)$, $A_4(\frac{1}{5}, \frac{2}{5}, 0, 0, \frac{2}{5})$ and $A_4(\frac{3}{5}, \frac{1}{5}, 0, 0, \frac{1}{5})$ do not have a rational solution of Type C. Therefore $A_4(\alpha_i)_{0 \leq i \leq 4}$ has a rational solution of Type C if and only if for some $i = 0, 1, \dots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) \equiv \begin{cases} \frac{j}{5}(1, 1, 1, 1, 1) & \text{mod } \mathbb{Z} \\ \frac{j}{5}(1, 2, 1, 3, 3) & \text{mod } \mathbb{Z}, \end{cases}$$

with some $j = 1, 2, 3, 4$. Furthermore, the rational solution is unique and can be transformed into

$$(f_0, f_1, f_2, f_3, f_4) = (\frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}) \text{ with } (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}).$$

We complete the proof of the main theorem.

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